Notes on Lectures 1 to 3

Basic Starting Point:

This course is about a "semi-rigorous" approach to Calculus. In Calculus, or more precisely "infinitesimal calculus", one aims to study certain kinds of functions.

So, functions will be the main objects to be studied.

Def. 1

Let D, E be two non-empty sets (a set is a collection of objects, usually denoted by enclosing those objects between two "curly" brackets, e.g. {1,2,3,4}. But often times, there are too many objects, and we are not able to "list" them all. In such cases, we we'll write {x | x satisfies some property} to define the set. E.g. {x | x is a real no. larger than 0} or equivalently, {x | x is a real no. & 0 < x}.), a function $f$ from D to E is a rule which sends each element in D to a unique element $f(x)$ in E.

Terminologies and Notations

If we are willing to use abstract mathematical notations, then we can use the symbol $\forall$ to mean "for each", "for all", "for every" or "for any".
We will also use the symbol $\exists$ to mean "there exists", or "for some".
(If we want to emphasize "there exists one and only one" (or "there exists unique"), we use the symbol $\exists!$)

Using the above notations, our Def. 1 takes the following form:

A function $f$ from D to E (D, E non-empty sets) is a rule satisfying:

$$\forall x \in D \; \exists! \, y \in E \mid y = f(x)$$

Comments:

During the lectures, I gave three pictures to visualize a function.

Terminologies:
1. D is called the domain of the function $f$, E is called the co-domain of $f$.
2. We will often denote a function by one letter (or more than one letter) such as $f$, $g$, $h$ etc. or $sin, cos, tan$.   (we may not write it in the form $f(x), g(x), h(x), sin(x),$

*cos(x), tan(x).* Why we do this is because we are thinking of the functions as "curves").

3.  We will often distinguish a function *f* from its evaluation *f(x).* (a function is a <u>rule</u> (or "instruction"), the "evaluation" is the "application of the <u>rule *f*</u> to the number *x.*)

Now, we give some examples of well-known functions.

First we have the

## Polynomial Functions (or simply "polynomials")

They are expressions+ of the form $p_n(x) \overset{\text{def}}{=} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$, where $a_0, a_1, a_2, \ldots, a_n$ are constants (i.e. "numbers") and $a_n \neq 0$ attached to the terms $x^1, x^2, \ldots, x^n$. These constants are known as coefficients of the terms $x^1, x^2, \ldots, x^n$ respectively.

Properties:

1.  <u>The equation $p_n(x) \overset{\text{def}}{=} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 = 0$ has at most *n* real number solutions.</u> (e.g. In school math, we learned that $p_1(x) = a_1 x + a_0 = 0$ which has either one solution or no solution. Similarly, $p_2(x) = a_2 x^2 + a_1 x + a_0 = 0$ has either no solution, one solution or two solutions.)

    The result underlined above is known as the <u>Fundamental Theore m of Algebra</u>.

2.  Two questions one often asks about a function such as the polynomial function $p_n(x)$ is (i) what is the maximal domain of it? (ii) what happens to $p_n(x)$ when x becomes very very large (in symbols, $x \to +\infty$) or becomes very very negative (in symbols, $x \to -\infty$). (Answers: (i) The maximal domain is always the set of all real nos., denoted by $\{x \mid -\infty < x < \infty\}$, (or by the symbol IR , or by the symbol $(-\infty, \infty)$.)

Comments:

In some future lectures, we will use the following short-form to say (i) and (ii) in Point 2 above, i.e. (i) will be abbreviated as $\lim_{x \to \infty} p_n(x) =?$ And (ii) will be abbreviated by

$$\lim_{x \to -\infty} p_n(x) =?$$

A polynomial function is a finite object, hence it is easy to handle by a computer.

The second simplest class of functions are

## Sine, Cosine Functions

Traditionally, they are defined respectively by

1. $\sin x = \frac{opposite\ side}{hypotenuse}$

2. $\cos x = \frac{adjacent\ side}{hypotenuse}$

Where one considers a right-angled triangle with angle $x$ between 0 and 90 degrees.

Comments:
1. In university mathematics, we will use "radians" rather than "degrees" to measure angles.
2. The above definition doesn't work for angle over 90 degrees, there one has to introduce "new" rules. Also when x goes beyond 360 degrees, one has to introduce another new rule. (In the literature, this kind of "introducing new rules to extend the domain of a function is known as analytic continuation". More about this later!)

Point 1 and Point 2 both reveal the fact that it is not too convenient to "write a program to compute" (or better "implement") the sine and cosine functions on a computer, if we try to use the right-angled triangle definition. One way to overcome this difficulty is to use another definition, which doesn't make use of triangle, i.e.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Comments:
1. In both cases in the above definitions, the right-hand sides are "infinite polynomials in $x$" (in later lectures, we will call such things "power series in $x$").

2. Because of the infinite lengths, it is worth thinking about how a computer can compute, say *sin(1.3)*? The difficulty lies in the fact that a computer program can only compute mathematical expressions that are finitely long, for otherwise the program would not stop!

**Addition Formula**

The following formula can be proved using trigonometry (done in schools) for angle *x* between 0 and 90 degrees. We, however, will give another proof.

$$\sin(u + v) = \sin(u)\cos(v) + \sin(v)\cos(u)$$
$$\cos(u + v) = \cos(u)\cos(v) - \sin(v)\sin(u)$$

Comment: You have to remember this formula, because it is of vital importance!

To prove it without any pictures of triangles, we need to introduce one more function, i.e. the

## Exponential Function

This is the function defined by

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Comment: One can show that the right-hand side is always a finite number, no matter what real no. *x* one inputs into it.

Now we prove the Addition Formulas.

First, let us do the following outrageous thing, i.e. to write down $\exp(\sqrt{-1} \cdot x)$.

This gives (you have to trust that this can be done!),

$$\exp(\sqrt{-1} \cdot x) = 1 + \frac{\sqrt{-1}\cdot x}{1!} + \frac{(\sqrt{-1}\cdot x)^2}{2!} + \frac{(\sqrt{-1}\cdot x)^3}{3!} + \frac{(\sqrt{-1}\cdot x)^4}{4!} + \frac{(\sqrt{-1}\cdot x)^5}{5!} + \cdots$$

Simplifying the right-hand side, we get

$$\exp(\sqrt{-1} \cdot x) = 1 + \frac{\sqrt{-1} \cdot x}{1!} - \frac{x^2}{2!} - \frac{\sqrt{-1} \cdot x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{\sqrt{-1} \cdot x}{1!} - \frac{\sqrt{-1} \cdot x^3}{3!} + \frac{\sqrt{-1} \cdot x^5}{5!}$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \sqrt{-1} \cdot \left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \cos(x) + \sqrt{-1} \cdot \sin(x)$$

Comment: This formula $\exp(\sqrt{-1} \cdot x) = \cos(x) + \sqrt{-1} \cdot \sin(x)$ is known as Euler's Formula.

Now we apply this to show the Addition Formula.

Let $x = u + v$ the we have $\exp\left(\sqrt{-1} \cdot (u + v)\right) = \cos(u + v) + \sqrt{-1} \cdot \sin(u + v)$

On the other hand, the left-hand side is

$$\exp\left(\sqrt{-1} \cdot (u + v)\right) = \exp(\sqrt{-1} \cdot u)\exp(\sqrt{-1} \cdot v)$$

$$= \left(\cos(u) + \sqrt{-1} \cdot \sin(u)\right) \cdot (\cos(v) + \sqrt{-1} \cdot \sin(v))$$

$$(\cos(u)\cos(v) - \sin(u)\sin(v)) + \sqrt{-1} \cdot (\sin(u)\cos(v) + \sin(v)\cos(u))$$

Comparing the terms <u>not</u> involving $\sqrt{-1}$ , we obtain the formula
$$\cos(u + v) = \cos(u)\cos(v) - \sin(u)\sin(v)$$

Comparing the terms involving $\sqrt{-1}$ , we obtain
$$\sin(u + v) = \sin(u)\cos(v) + \sin(v)\cos(u).$$

Related to the exponential function is the

## Logarithm Function
or Log Function, defined by

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

<u>Comment:</u>

On the left-hand side of the "=" sign, we have 1+x instead of x inside the bracket!

We can, if we want, change it back to the form ln(x). To do this, let u=1+x and obtain

$$\ln(u) = (u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \cdots$$

Changing the symbol for the variable from $u$ back to $x$, we obtain

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

<u>Question:</u> What is the maximal domain of the exponential function ? of the log function ?

Having introduced the sine, cosine, exponential and the log functions, we can now describe a relationship between the exponential and the log function.

## Exponential Function, Log Function, Inverse Function

In school math, we learned that $10^{\log_{10} x} = x$, where $x$ is any positive real no.

We also know that for any positive real no. $a$, we have

$$a^{\log_a x} = x$$

Now we can take $a$ to be the number defined by

$$e = \exp(1) = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.7 \ldots$$

Using this real no. as "base", we have (this can be shown to be the case!) the

logarithm based on *e* , of any positive real number, denoted by ln(x) (also known as "natural logarithm" or "Napier logarithm") given by the formula

$$e^{\ln(x)} = x.$$

<u>Comment:</u>

This formula means that exp(x) (or $e^x$) is the inverse function of ln(x) (and vice versa). (Here I wrote exp(x), with the variable attached because this way, it would easier to understand than just exp).

With a lot of work, one can show that ln(x) can be written in the form

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

<u>Comments:</u>

There is one more way to define exp(x), where *x* is any real no. (positive, zero or negative).

This is

$$\exp(x) = \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n.$$

One can show that

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n$$